

# KÄHLER PACKINGS AND SESHADRI CONSTANTS ON PROJECTIVE COMPLEX SURFACES

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**ABSTRACT.** In analogy to the relation between symplectic packings and symplectic blow ups we show that multiple point Seshadri constants on projective complex surfaces can be calculated as the supremum of radii of multiple Kähler ball embeddings.

Symplectic Topology, searching for global properties of symplectic manifolds, is a rather new branch of the old and venerable study of symplectic structures (see the in-depth treatise of McDuff and Salamon, [MS95]). One of its most striking successes is the analysis and solution of several symplectic packing problems: How large can symplectic balls of a given number be when disjointly embedded in a given symplectic manifold? These questions are especially attractive as they exhibit the fundamental nature of symplectic structures: local flexibility vs (sometimes) global strictness. In particular, symplectic packing is not so strict as Euclidean (that is, distance-and-angle preserving) packing, leading to questions like the Kepler conjecture on ball packings (and its solution by Hales [Hal05, Hal12]), but may be not so flexible as only volume-preserving packing. So some symplectic packing problems reveal obstacles to packings without gaps, whereas other packings are possible without gaps.

One of the most prominent series of such symplectic packing problems asks how large symplectic balls of a given number can be when disjointly embedded in the complex projective plane  $\mathbb{CP}^2$ . In more details, consider balls  $B_0(r) \subset \mathbb{R}^4$  of radius  $r$  centered in 0 together with the symplectic form  $\omega_{\text{std}}$  obtained by restricting the standard symplectic form on  $\mathbb{R}^4$ . If  $\coprod_{q=1}^k B_0(r_q)$  denotes the disjoint union of  $k$  of these balls, with possibly different radii, and  $\omega_{\text{FS}}$  denotes a Fubini-Study Kähler form on  $\mathbb{CP}^2$ , then a symplectic packing of  $\mathbb{CP}^2$  with  $k$  symplectic balls is defined as a symplectic embedding

$$\iota : \coprod_{q=1}^k B_0(r_q) \hookrightarrow \mathbb{CP}^2,$$

that is,  $\iota$  is a smooth embedding such that  $\iota^* \omega_{\text{FS}}|_{B_0(r_q)} = \omega_{\text{std}}$ . The symplectic packing problem asks on conditions on the radii  $r_q$  such that such a symplectic packing exist, and also how it can be explicitly constructed.

McDuff and Polterovich [MP94] connected this problem first to symplectic blow ups and then to Algebraic Geometry: They showed that a symplectic packing with balls of radii  $r_q$  is only possible if on  $\sigma : X \rightarrow \mathbb{CP}^2$ , the blow up of  $\mathbb{CP}^2$  (considered as a complex manifold) in  $k$  points  $x_1, \dots, x_k$ , there exists a symplectic form representing

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the cohomology class of  $\sigma^*l - \pi \sum_{q=1}^k r_q^2 e_q$ , where  $l$  and the  $e_q$  are Poincaré dual to a line  $L \subset \mathbb{CP}^2$  and the exceptional divisors  $E_q = \sigma^{-1}(x_q)$ . Then they proved that as long as  $k \leq 8$  the only obstacles to the existence of such a symplectic form are the same as for the existence of a Kähler form in this cohomology class, namely  $(-1)$ -curves on  $X$ . Thus, these symplectic packing problems are merged with the algebraic-geometric theory of del Pezzo surfaces already extensively studied in the 19th century (see [Man74] for a survey and results).

Next, Biran [Bir97] was able to prove that  $(-1)$ -curves remain the only obstacles for symplectic packings with  $k \geq 9$  balls, and as a consequence he showed the symplectic analogue of a celebrated algebraic-geometric conjecture named after Nagata, who came across it when solving Hilbert's Fourteenth Problem [Nag59].

**Conjecture 1** (Nagata). *With notations as above and  $k \geq 9$ , there is a Kähler form representing the cohomology class  $\sigma^*l - \epsilon \cdot \sum_{q=1}^k e_q$  for all  $\epsilon < \frac{1}{\sqrt{k}}$  if the blown-up points  $x_1, \dots, x_k$  are chosen sufficiently general.*

Note that this is the Kähler version of a purely algebraic-geometric statement:

**Conjecture 2** (Nagata, algebraic-geometric version). *Let  $C = \{F = 0\}$  be an irreducible algebraic curve in  $\mathbb{CP}^2$ , given by an irreducible homogeneous polynomial  $F = F(X, Y, Z)$  of degree  $d$  in three homogeneous variables  $X, Y, Z$  and with multiplicity  $m_q$  in the point  $x_q$  (that is,  $m_q$  is the lowest degree of a non-vanishing term in the Taylor series expansion of  $F$  around  $x_q$ ). If the points  $x_1, \dots, x_k \in \mathbb{CP}^2$  are chosen sufficiently general then*

$$\sqrt{k}d \geq \sum_{q=1}^k m_q.$$

The equivalence of these two conjectures follows from the fact that Kähler forms representing an integral cohomology class are curvature forms of hermitian metrics on an ample line bundle (that is Kodaira's embedding theorem [Wel80, Thm.III.4.1, III.4.6]) and that intersection numbers of ample cohomology classes with algebraic curves are always positive (that is the easy half of Nakai-Moishezon's Ampleness Criterion [Laz04, Thm.1.2.23]). The Nakai-Moishezon Criterion is applied on the strict transform  $\overline{C}$  on  $X$ , that is the inverse image of  $C$  under the blow-up map  $\sigma$  without the exceptional divisors  $E_q$  (intersected  $m_q$  times by  $\overline{C}$ ).

More on Biran's proof and on what the symplectic methods tell us for the algebraic situation (in particular, why they cannot be used so easily for Nagata's Conjecture) can be found in Biran's lucid survey [Bir01].

The aim of this note is to show that the algebraic conjecture of Nagata is equivalent to a more restricted packing problem, namely a *Kähler packing problem*.

Before making an exact statement, let us introduce some notation:

**Definition 3.** *Let  $(V, \Omega)$  be a  $n$ -dimensional Kähler manifold with Kähler form  $\Omega$ . Then a holomorphic embedding*

$$\phi = \coprod_{q=1}^k \phi_q : \coprod_{q=1}^k B_0(r_q) \rightarrow V$$

*is called a Kähler embedding of  $k$  disjoint standard complex balls in  $\mathbb{C}^n$  centered in 0, of radii  $r_q$ , if  $\phi_q^*(\Omega) = \omega_{\text{std}}$ , the standard Kähler form on  $\mathbb{C}^n$  restricted to  $B_0(r_q)$ .*

From now on, let  $V$  be a 2-dimensional projective complex manifold,  $x_1, \dots, x_k \in V$  distinct points on  $V$  and  $L$  an ample line bundle on  $V$ . Sometimes we interpret  $L$  also as a divisor on  $V$ . Let  $\sigma : \tilde{V} \rightarrow V$  be the blow up of the  $k$  points  $x_1, \dots, x_k$ , with exceptional divisors  $E_q = \sigma^{-1}(x_q)$ .

**Definition 4.** *The multi-point Seshadri constant  $\epsilon(L; x_1, \dots, x_k)$  is defined as*

$$\sup\{\epsilon > 0 : \text{A multiple of } \sigma^*L - \epsilon \sum_{q=1}^k E_q \text{ is an ample divisor}\}.$$

Seshadri constants are busily investigated in Algebraic Geometry during the last years, as a measure of local positivity (see e.g. [Laz04, Ch.5]).

**Theorem 5.** *For all  $0 < \epsilon < \epsilon(L; x_1, \dots, x_k)$  there exists a Kähler form  $\omega$  on  $V$  representing the first Chern class  $c_1(L)$  of  $L$  and a Kähler packing (wrt  $\omega$ ) of  $k$  disjoint balls of radii  $\sqrt{\frac{\epsilon}{\pi}}$  into  $V$ . Vice versa, if for  $\epsilon > 0$  there exists such a Kähler packing wrt a Kähler form  $\omega$  representing  $c_1(L)$  then  $\epsilon < \epsilon(L; x_1, \dots, x_k)$ .*

*Proof.* The idea to prove the first direction of the theorem is to construct Kähler metrics on  $V$  from sections of  $L^{\otimes m}$  whose corresponding Kähler forms represent  $c_1(L)$  and which concentrate more and more volume around the points  $x_1, \dots, x_k$ . If the sections are carefully chosen this concentration proceeds in a homogeneous way, and the Kähler metrics get sufficiently flat around the  $x_q$  to be able to glue in a standard Kähler ball of a radius arbitrarily close to  $\sqrt{\frac{\epsilon}{\pi}}$ . The main technical tool for the gluing is the symplectic blow down described by McDuff and Polterovich [MP94, §5.4].

In more details, recall that the standard Kähler form  $\omega_0$  on  $\mathbb{C}^2$  is given in affine holomorphic coordinates  $(x, y)$  by  $\frac{i}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y})$ , whereas the Fubini-Study Kähler form  $\tau_0$  on  $\mathbb{CP}^1$  is given in homogeneous coordinates  $[S : T]$  by  $\frac{i}{2\pi} \partial \bar{\partial} \log(S\bar{S} + T\bar{T})$ . Note that the latter  $(1, 1)$ -form is well-defined on  $\mathbb{CP}^1$  because  $S\bar{S} + T\bar{T}$  is homogeneous in  $S$  and  $T$ , and that it represents  $c_1(\mathcal{O}_{\mathbb{CP}^1}(P))$  for any point  $P \in \mathbb{CP}^1$ . More generally, if  $s_0, \dots, s_N$  are sections of a line bundle  $L$  on a complex manifold  $X$  defining an embedding

$$X \hookrightarrow \mathbb{CP}^N, \quad x \mapsto [S_0 : \dots : S_N] = [s_0(x) : \dots : s_N(x)]$$

(for example, if the  $s_i$  span  $H^0(X, L)$  and  $L$  is very ample) then we can use this embedding to construct a Kähler form on  $X$ , by restricting the Fubini-Study form  $\frac{i}{2\pi} \partial \bar{\partial} \log(\sum_{k=0}^N S_k \bar{S}_k)$  in homogeneous coordinates  $[S_0 : \dots : S_N]$  on  $\mathbb{CP}^N$  to  $X$ . We say that this restricted Kähler form on  $X$  is *induced* by the sections  $s_0, \dots, s_N$ . The Kähler form can also be seen as the curvature form of the hermitian metric  $h$  induced by the sections on  $L$ , defining the length of the vector  $\xi(x)$  for each section  $\xi$  of  $L$  and point  $x$  on  $X$  by

$$\|\xi\|_h^2 := \frac{\xi(x) \overline{\xi(x)}}{\sum_{k=0}^N s_k(x) \overline{s_k(x)}}.$$

Let  $(x, y)$  be local complex coordinates around  $x_q \in V$ , and denote by  $S := \frac{x}{y}, T := \frac{y}{x}$  the induced homogeneous coordinates on the exceptional divisor  $E_q \cong \mathbb{CP}^1$ . If  $U_q(\delta)$  denotes a ball centered in  $x_q$  of sufficiently small radius  $\delta$ , measured according to the coordinates  $x, y$ , then the tubular neighborhood

$\sigma^{-1}(U_q(\delta)) \subset \tilde{V}$  of  $E_q$  is projected to  $E_q \cong \mathbb{CP}^1$  by a holomorphic map  $p_q$  collapsing the lines in  $U_q(\delta)$  through  $(0,0)$ . Furthermore  $\sigma^{-1}(U_q(\delta))$  is covered by two charts with coordinates  $(x,t)$  resp.  $(s,y)$ , with transition maps given by  $y = xt$  and  $s = 1/t$ . Note that the exceptional divisor  $E_q$  intersects these charts as the vanishing locus of  $x$  resp.  $y$ .

Now assume that  $\epsilon \in \mathbb{Q}$ . Then for  $n > 0$  a sufficiently divisible integer, the line bundle  $\tilde{L}_n := \sigma^*(L^{\otimes n}) \otimes \mathcal{O}_{\tilde{V}}(-n\epsilon \cdot \sum_{q=1}^k E_q)$  is ample. On  $U_q(\delta)$  the line bundle  $L^{\otimes n}$  is trivial, hence we can define a hermitian metric  $h_0$  on  $L^{\otimes n}|_{U_q(\delta)}$  by

$$\|\xi\|_{h_0}^2 := \frac{\xi(x)\overline{\xi(x)}}{e^{x\bar{x}+y\bar{y}}}$$

with everywhere positive curvature form  $\frac{1}{\pi}\omega_0 = \frac{i}{2\pi}(dx \wedge d\bar{x} + dy \wedge d\bar{y})$ . If  $\sigma^*h_0$  denotes the pulled back metric on  $\sigma^*(L^{\otimes n})$  its curvature form  $\sigma^*\omega_0$  is everywhere semipositive on  $U_q(\delta)$  and positive away from  $E_q$ .

The sections of  $\mathcal{O}_{\sigma^{-1}(U_q(\delta))}(-n\epsilon \cdot \sum_{q=1}^k E_q)$  given in the coordinates  $(x,t)$  of one of the charts covering  $\sigma^{-1}(U_q(\delta))$  by

$$\sqrt{\binom{n\epsilon}{j}} t^j, \quad j = 0, \dots, n\epsilon,$$

define a hermitian metric  $h_q$  on  $\mathcal{O}_{\sigma^{-1}(U_q(\delta))}(-n\epsilon \cdot \sum_{q=1}^k E_q)$ . Note that the coefficient of  $t^j$  allows to rewrite the metric induced by these sections on  $\mathcal{O}_{\sigma^{-1}(U_q(\delta))}(-n\epsilon \cdot \sum_{q=1}^k E_q)$  as a power of the metric induced by the sections 1 and  $t$  on  $\mathcal{O}_{\sigma^{-1}(U_q(\delta))}(-\sum_{q=1}^k E_q)$ . Since the curvature form of  $h_q$  is positive on  $E_q$  the tensor product  $\sigma^*h_0 \otimes h_q$  is a hermitian metric  $h_{0,q}$  on  $\tilde{L}_n|_{\sigma^{-1}(U_q(\delta))}$  with everywhere positive curvature form

$$\omega_{0,q} := \frac{1}{\pi}\sigma^*\omega_0 + n\epsilon \cdot p_q^*\tau_0.$$

*Step 1.* For  $n \gg 0$  we can find sections  $\sigma_0, \dots, \sigma_N$  spanning  $H^0(\tilde{V}, \tilde{L}_n)$  such that the induced hermitian metric  $\tilde{h}$  on  $\tilde{L}_n$  has a positive curvature form  $\tilde{\omega}$  satisfying

$$\tilde{\omega}|_{E_q} = n\epsilon \cdot \tau_0 \text{ and } \tilde{\omega}(P) = \frac{1}{\pi}(\sigma^*\omega_0)(P) + n\epsilon \cdot (p_q^*\tau_0)(P),$$

for all  $q = 1, \dots, k$  and all points  $P \in E_q$ : For  $n \gg 0$  the line bundle  $\tilde{L}_n$  is sufficiently ample such that the restriction maps

$$\begin{aligned} H^0(\tilde{V}, \tilde{L}_n \otimes \mathcal{O}_{\tilde{V}}(-2 \sum_{r=1}^k E_r)) &\rightarrow \bigoplus_{r=1}^k H^0(E_r, \tilde{L}_n|_{E_r} \otimes \mathcal{O}_{E_r}(-2E_r)) = \\ &= \bigoplus_{r=1}^k H^0(E_r, \mathcal{O}_{E_r}(-(n\epsilon + 2)E_r)), \\ H^0(\tilde{V}, \tilde{L}_n \otimes \mathcal{O}_{\tilde{V}}(-\sum_{r=1}^k E_r)) &\rightarrow \bigoplus_{r=1}^k H^0(E_r, \mathcal{O}_{E_r}(-(n\epsilon + 1)E_r)), \\ H^0(\tilde{V}, \tilde{L}_n \otimes \mathcal{O}_{\tilde{V}}(-\sum_{r \neq q} E_r)) &\rightarrow \bigoplus_{r \neq q} H^0(E_r, \mathcal{O}_{E_r}(-(n\epsilon + 1)E_r)) \oplus H^0(E_q, \mathcal{O}_{E_q}(-n\epsilon E_q)) \end{aligned}$$

are surjective for each  $q = 1, \dots, k$ , by Serre Vanishing [Laz04, Thm.1.2.6]. For each  $q = 1, \dots, k$  we can thus find

- sections in  $H^0(\tilde{V}, \tilde{L}_n)$  restricting to  $S^{n\epsilon+2}, S^{n\epsilon+1}T, \dots, T^{n\epsilon+2}$  on  $E_q$  (when divided by the square of the defining function of  $E_q$ ) and vanishing to order  $\geq 2$  on all exceptional divisors  $E_r \neq E_q$ ,
- sections in  $H^0(\tilde{V}, \tilde{L}_n)$  restricting to (scalar multiples of)  $S^{n\epsilon+1}, S^{n\epsilon}T, \dots, T^{n\epsilon+1}$  on  $E_q$  (when divided by the defining function of  $E_q$ ) and vanishing to order  $\geq 2$  on all exceptional divisors  $E_r \neq E_q$ , and
- sections in  $H^0(\tilde{V}, \tilde{L}_n)$  restricting on  $E_q$  to (scalar multiples of)  $S^{n\epsilon}, S^{n\epsilon-1}T, \dots, T^{n\epsilon}$ , a basis of  $H^0(E_q, \mathcal{O}_{E_q}(-n\epsilon E_q))$  and vanishing to order  $\geq 2$  on all exceptional divisors  $E_r \neq E_q$ .

Uniting a basis of  $H^0(\tilde{V}, \tilde{L}_n \otimes \mathcal{O}_{\tilde{V}}(-2 \sum_{r=1}^k E_r)) \subset H^0(\tilde{V}, \tilde{L}_n)$  with suitable linear combinations of the sections above we obtain a set of sections  $\sigma_0, \dots, \sigma_N$  spanning  $H^0(\tilde{V}, \tilde{L}_n)$  which can be subdivided in three disjoint parts for each  $q = 1, \dots, k$ : In terms of the  $(x, t)$  coordinates in one of the charts around  $E_q$  the sections are either of the form

$$\sqrt{\binom{n\epsilon}{j}} t^j + x^3 \cdot f_j(x, t), \quad j = 0, \dots, n\epsilon, \text{ or}$$

$$x \cdot \left( \sqrt{\binom{n\epsilon+1}{l}} t^l + x \cdot g_l(x, t) \right), \quad l = 0, \dots, n\epsilon + 1, \text{ or } x^2 \cdot h(x, t),$$

where the  $f_j, g_l$  and  $h$  are regular functions in  $x, t$  and there exists exactly one section of the respective form for each  $j$  and each  $l$ . By multiplying with  $s^{n\epsilon}$  and using  $t \cdot s = 1$  and  $x = sy$  we obtain expressions for the sections in the  $(s, y)$ -coordinates of the other chart around  $E_q$ , and these expressions in  $s, y$  are completely similar to those in  $x, t$ .

Let  $\tilde{h}$  denote the hermitian metric on  $\tilde{L}_n$  and  $\tilde{\omega}$  the Kähler form on  $\tilde{V}$  induced by the sections  $\sigma_0, \dots, \sigma_N$ . Using the coordinates  $(x, t)$  around  $E_q$  (the calculations are completely analogous when using the coordinates  $(s, y)$  of the other chart around  $E_q$ ) the Taylor series expansion of log shows that

$$\begin{aligned} \tilde{\omega}(0, 0) &= \left( \frac{i}{2\pi} \partial \bar{\partial} \log \left( \sum_{j=0}^N \sigma_j(x, t) \overline{\sigma_j(x, t)} \right) \right) (0, 0) = \frac{1}{\pi} (dx \wedge d\bar{x} + n\epsilon dt \wedge d\bar{t}) = \\ &= \frac{1}{\pi} \sigma^* \omega_0(0, 0) + n\epsilon p_2^* \tau_0(0, 0), \end{aligned}$$

because  $F := \sum_{j=0}^N \sigma_j(x, t) \overline{\sigma_j(x, t)}$  is a power series in  $x, \bar{x}, t, \bar{t}$ , and the only terms of order  $\leq 2$  in  $F$  are  $1, x\bar{x}, n\epsilon t\bar{t}$ . Similarly in other points  $P = (0, t_0) \in E_q$ : Choose a unitary matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2)$  such that  $t_0 = \frac{c}{a}$ , and rewrite  $F$  in terms of the coordinates  $(x', t')$  given by

$$x = x'(a + bt'), \quad t = \frac{c + dt'}{a + bt'}.$$

Then  $F \cdot |a + bt'|^{2n\epsilon}$  is a power series in  $x', \bar{x}', t', \bar{t}'$ , and as before the only terms of order  $\leq 2$  are  $1, x'\bar{x}', n\epsilon t'\bar{t}'$  because  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2)$  implies that

$$(1 + t\bar{t}) \cdot |a + bt'|^2 = |a + bt'|^2 + |c + dt'|^2 = 1 + t'\bar{t}'.$$

Since  $\partial\bar{\partial} \log |a + bt'|^2 = 0$  we conclude once again that

$$\tilde{\omega}(P) = \frac{1}{\pi} dx' \wedge d\bar{x}' + n\epsilon dt' \wedge d\bar{t}' = \frac{1}{\pi} \sigma^* \omega_0(P) + n\epsilon p_2^* \tau_0(P).$$

Finally, the statement on  $\tilde{\omega}|_{E_q}$  follows because  $F|_{x=0} = (1 + t\bar{t})^{n\epsilon}$ .

*Step 2.* On the tubular neighborhoods  $\sigma^{-1}(U_q(\delta))$  we glue the metrics  $\tilde{h}$  on  $\tilde{L}_n$  and  $h_{0,q}$  on  $\tilde{L}_n|_{\sigma^{-1}(U_q(\delta))}$ : To this purpose we use a partition of unity  $(\bar{\rho}_1, \bar{\rho}_2)$  subordinate to the open cover  $(\tilde{V} - \sigma^{-1}(U_q(\delta/2)), \sigma^{-1}(U_q(\delta)))$  of  $\tilde{V}$ . We construct  $\bar{\rho}_1, \bar{\rho}_2$  from a partition of unity  $(\rho_1, \rho_2)$  subordinate to the open cover  $(\mathbb{R} - (-\delta^2/4, \delta^2/4), (-\delta^2, \delta^2))$  of  $\mathbb{R}$ , by setting

$$\bar{\rho}_i(x, t) := \rho_i(|\sigma(x, t)|^2) = \rho_i(|\sigma(s, y)|^2), i = 1, 2.$$

Note that we can choose  $\rho_i$  such that the first-order partial derivatives of  $\bar{\rho}_i$  are bounded by a constant multiple of  $1/\delta$  and the second-order partial derivatives of  $\bar{\rho}_i$  by a constant multiple of  $1/\delta^2$ .

Let  $\sigma_0, \dots, \sigma_N$  be the global sections of  $\tilde{L}_n$  constructed in Step 1. Then in coordinates  $(x, t)$  around  $E_q$  the metric  $\tilde{h}$  induced by these sections is given by

$$\tilde{h}(\sigma(x, t)) = |\sigma(x, t)| / \left( \sum_{j=0}^N |\sigma_j(x, t)|^2 \right)^{\frac{1}{2}} = |\sigma(x, t)| \cdot e^{-\frac{1}{2}\phi_1(x, t)},$$

with  $\phi_1(x, t) = \log(\sum_{j=0}^N |\sigma_j(x, t)|^2)$ , for each section  $\sigma$  of  $\tilde{L}_n$ . Similarly,

$$h_{0,q}(\sigma(x, t)) = |\sigma(x, t)| \cdot e^{-\frac{1}{2}\phi_2(x, t)}$$

with  $\phi_2 = n\epsilon \log(1 + t\bar{t}) + \frac{1}{\pi} x\bar{x}(1 + t\bar{t})$ . Then the glued metric  $\bar{h}$  can be constructed as

$$\bar{h}(\sigma(x, t)) = |\sigma(x, t)| e^{-\frac{1}{2}(\bar{\rho}_1 \phi_1 + \bar{\rho}_2 \phi_2)} = |\sigma(x, t)| e^{-\frac{1}{2}(\phi_1 + \bar{\rho}_2(\phi_2 - \phi_1))}.$$

Its curvature is

$$\begin{aligned} \frac{i}{2\pi} [\partial\bar{\partial}\phi_1 + \partial\bar{\partial}(\bar{\rho}_2(\phi_2 - \phi_1))] &= \frac{i}{2\pi} [\partial\bar{\partial}\phi_1 + \partial(\bar{\rho}_2 \cdot \bar{\partial}(\phi_2 - \phi_1) + \bar{\partial}\bar{\rho}_2 \cdot (\phi_2 - \phi_1))] = \\ &= \frac{i}{2\pi} [\partial\bar{\partial}\phi_1 + \partial\bar{\rho}_2 \cdot \bar{\partial}(\phi_2 - \phi_1) + \bar{\rho}_2 \cdot \partial\bar{\partial}(\phi_2 - \phi_1) + \partial\bar{\partial}\bar{\rho}_2 \cdot (\phi_2 - \phi_1) + \bar{\partial}\bar{\rho}_2 \cdot \partial(\phi_2 - \phi_1)]. \end{aligned}$$

The Taylor series expansion of  $\log$  and the properties of the sections  $\sigma_j$  discussed in Step 1 show that  $\phi_2 - \phi_1$  expands to a power series in  $x, t$  only containing terms of order  $\geq 3$ . Hence the remarks on the partial derivatives of  $\bar{\rho}_1$  and  $\bar{\rho}_2$  imply that around  $(x, t) = (0, 0)$  all summands but the first converge everywhere on  $\sigma^{-1}(U_q(\delta)) - \sigma^{-1}(U_q(\delta/2))$  to 0 when  $\delta$  tends to 0. Since  $\frac{i}{2\pi} \partial\bar{\partial}\phi_1$  is strictly positive being the curvature of  $\tilde{h}$ , it follows that  $\bar{h}$  is a positive metric on  $\tilde{L}^n$  for  $n$  sufficiently large and  $\delta$  sufficiently small. Calling  $\bar{\omega}$  the Kähler form obtained as the curvature of  $\bar{h}$  we have that

$$\bar{\omega}|_{\sigma^{-1}(U_q(\delta/2))} = \omega_{0,q}.$$

*Step 3.* We glue in standard Kähler balls of radius  $\sqrt{\frac{\epsilon}{\pi}}$  to  $(\tilde{V}, \frac{1}{n}\bar{\omega})$  replacing the exceptional divisors  $E_q$ : Let  $\mathcal{L}(r)$  denote the preimage of the ball  $B(r)$  centered in

$0 \in \mathbb{C}^2$  under the standard blow-up  $\sigma$  of  $\mathbb{C}^2$  in 0 and let  $\rho(\delta, \epsilon)$  denote the Kähler form

$$\rho(\delta, \epsilon) := \delta \cdot \sigma^* \omega_0 + \epsilon \cdot p_2^* \tau_0$$

on  $\mathcal{L}(r)$ , for  $\delta, \epsilon > 0$ . The construction of  $\bar{\omega}$  implies that an appropriate rescaling of the  $(x, y)$ -coordinates around  $x_q$  without changing the homogeneous coordinates  $S, T$  on  $E_q$  yields holomorphic embeddings

$$\phi_q : \mathcal{L}(1 + \epsilon_q) \hookrightarrow \tilde{V}$$

such that  $\phi_q^*(\frac{1}{n}\bar{\omega}) = \rho(\delta_q, \epsilon)$ , for some  $\epsilon_q, \delta_q > 0$ . The symplectic blow-down construction in [MP94, §5.4, in particular §5.4.A] shows that there exist a Kähler form on  $V$  representing  $c_1(L)$  and a Kähler embedding of  $k$  standard balls of radii  $\sqrt{\frac{\epsilon}{\pi}}$  into  $V$ , wrt this Kähler form.

The opposite direction of the theorem follows immediately by using symplectic blow up constructions on Kähler manifolds as described in [MP94, §5.3, in particular §5.3.A].  $\square$

*Remark 6.* A proof that is just notationally more involved will show the analogous theorem for projective complex manifolds of arbitrary dimension and their multi-point Seshadri constants.

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